The Kirchhoff indices and the matching numbers of unicyclic graphs

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Abstract The Kirchhoff index of a connected graph is the sum of resistance distances between all unordered pairs of vertices in the graph. It found considerable applications in a variety of fields. In this paper, we determine the minimum Kirchhoff index among the unicyclic graphs with fixed number of vertices and matching number, and characterize the extremal graphs.

1 Introduction

The resistance distance was introduced by Klein and Randić [8] as a distance function on a graph. Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The resistance distance between vertices u and v of G, denoted by $r_G(u,v)$, is defined as the effective resistance between nodes u and v of the electrical network for which nodes correspond to the vertices of G and each edge of G is replaced by a resistor of unit resistance (one ohm).

The Kirchhoff index of a connected graph G is defined as [1]

$$Kf(G) = \sum_{\{u,v\}\subseteq V(G)} r_G(u,v).$$

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It is also named as total effective resistance [6]. This graph invariant found applications in chemistry, electrical network, Markov chains, averaging networks, experiment design, and Euclidean distance embeddings, see [7, 1, 6].

The (ordinary) distance between vertices u and v of a graph G, denoted by $d_G(u, v)$, is the length of a shortest path connecting them in G. Recall that the Wiener index of G is defined as [4, 5] $W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v)$. It has been shown [8] that $r_G(u, v) \leq d_G(u, v)$ with equality if and only if there is a unique path connecting u and v in G. As a consequence, the Kirchhoff index for a tree is equal to its Wiener index, which has been extensively studied (see [4]). Thus the Kirchhoff index is primarily of interest in the case of cycle-containing graphs.

Zhou and Trinajstić [13, 14] established various lower and upper bounds for the Kirchhoff index, see also [15]. Among the n-vertex connected graphs, Lukovits et al. [9] showed that the complete graph K_n is the unique graph with minimum Kirchhoff index, and Palacios [10] showed that the path P_n is the unique graph with maximum Kirchhoff index. The maximum and minimum Kirchhoff indices among the unicyclic graphs have been determined by Yang and Jiang [11], see also [12].

A matching M of the graph G is a subset of $E(G)$ such that no two edges in M share a common vertex. A matching M of G is said to be maximum, if for any other matching M' of G, $|M'| \leq |M|$. The matching number of G is the number of edges of a maximum matching in G. For a matching M of a graph G, if the vertex $v \in V(G)$ is incident with an edge of M, then v is said to be M-saturated. Moreover, if every vertex of G is M-saturated, then M is a perfect matching of G.

Zhou and Trinajstić [16] determined the graphs with minimum Wiener index and Kirchhoff index respectively among the connected graphs with fixed number of vertices and matching number. Du and Zhou [3] determined the graphs with minimum Wiener index among the trees and unicyclic graphs respectively with fixed number of vertices and matching number.

In this paper, we determine the minimum Kirchhoff index among the unicyclic graphs with fixed number of vertices and matching number, and characterize the extremal graphs. It is of interest to point out that among the unicyclic graphs with fixed number of vertices and matching number, the graphs with minimum Kirchhoff index are different from those with minimum Wiener index (see [3]).

2 Preliminaries and Lemmas

For a graph G with $v \in V(G)$, $G - v$ denotes the graph resulting from G by deleting v (and its incident edges). For an edge uv of the graph G (the complement of G, respectively), $G - uv$ ($G + uv$, respectively) denotes the graph resulting from G by deleting (adding, respectively) uv.

For $u \in V(G)$, let $Kf_G(u) = \sum\limits_{v \in V(G)} r_G(u,v)$. Then

$$Kf(G) = \frac{1}{2} \sum_{u \in V(G)} Kf_G(u).$$

Let C_n be the cycle on $n \geq 3$ vertices, whose vertices are labeled consecutively by $v_1, v_2, \ldots, v_n$.

For two vertices $v_i, v_j \in V(C_n)$ with $i < j$, by Ohm's law, we have

$$r_{C_n}(v_i, v_j) = \frac{(j-i) \cdot [n-(j-i)]}{n}. \tag{1}$$

Furthermore, for fixed n, $r_{C_n}(v_i, v_j)$ is increasing for $j - i \leq \lfloor \frac{n}{2} \rfloor$. For $v_1 \in V(C_n)$, by Eq. (1), we have

$$Kf_{C_n}(v_1) = \sum_{i=2}^{n} r_{C_n}(v_1, v_i) = \sum_{i=2}^{n} \frac{(i-1) \cdot [n-(i-1)]}{n} = \frac{n^2-1}{6}, \tag{2}$$

and thus

$$Kf(C_n) = \frac{1}{2} \cdot n \cdot Kf_{C_n}(v_1) = \frac{n^3-n}{12}. \tag{3}$$

For a unicyclic graph G with the unique cycle C_k, $G - E(C_k)$ consists of k vertex-disjoint trees $T_1, T_2, \ldots, T_k$, where $v_i \in V(T_i)$ for $i = 1, 2, \ldots, k$. These trees are called the branches of G, and v_i is called the root of the branch T_i in G for $i = 1, 2, \ldots, k$.

Now we define the graph $U(k,t,i,j)$ which will be used frequently later. For integers k, t, i, j with $k \geq 3$, $k \geq t \geq 0$, $i \geq 0$, $j \geq 0$, let $U(k,t,i,j)$ be the graph obtained from the cycle C_k as follows:

(a) choose t consecutive vertices in the cycle C_k;

(b) attach t pendent vertices each to one of the t chosen vertices in (a);

(c) attach i pendent vertices and j paths on two vertices to a central vertex of the t chosen vertices in (a).

Clearly, $U(k,t,i,j)$ has $k+t+i+2j$ vertices. In particular, let $U(k,t) = U(k,t,0,0)$ for integers k, t with $k \geq 3$ and $k \geq t \geq 0$. For example, $U(3,1,0,3)$, $U(3,2,2,1)$ and $U(3,3,1,1)$ are shown in Fig. 1.

Let $d_G(v)$ be the degree of v in G.

For integers n and m with $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$, let $\mathbb{U}(n,m)$ be the set of unicyclic graphs with n vertices and matching number m. For integer $m \geq 2$, we can partition $\mathbb{U}(2m,m) \setminus \{C_{2m}\}$ into two subsets as follows:

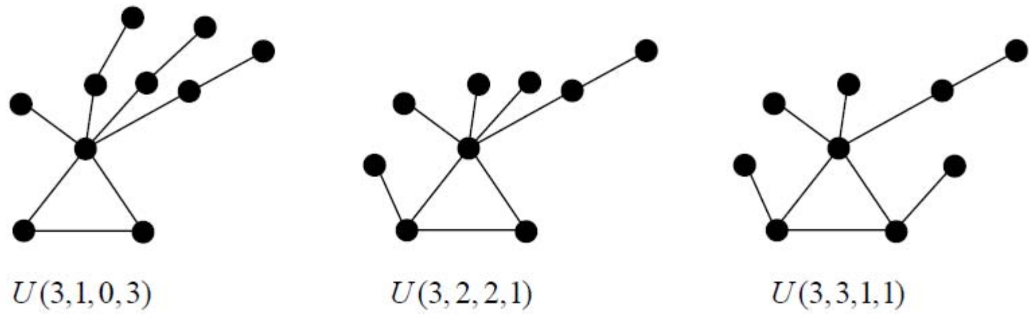

Figure 1: The graphs $U(3,1,0,3)$, $U(3,2,2,1)$ and $U(3,3,1,1)$.

(i) the set of graphs of maximum degree three in $\mathbb{U}(2m,m)$ obtainable by attaching some pendent vertices to a cycle, which is denoted by $\mathbb{U}_1(m)$;

(ii) the set of graphs in $\mathbb{U}(2m,m)$ containing some pendent vertex whose unique neighbor is of degree two, which is denoted by $\mathbb{U}_2(m)$.

2.1 The Kirchhoff index of graphs in $\mathbb{U}_1(m)$ with small m

First we want to determine the minimum Kirchhoff index among the graphs in $\mathbb{U}_1(m)$ with $2 \leq m \leq 8$.

Lemma 2.1. *Let $G \in \mathbb{U}_1(m)$ with the unique cycle C_k and t pendent vertices, where $k+t=2m$, $k \geq 3$ and $k \geq t \geq 1$.*

(i) *For $t = 1,2,3,k-4,k-2,k$, $(k,t)=(10,4)$, $(k,t)=(11,5)$, or $(k,t)=(12,4)$, we have*

$$Kf(G) \geq \frac{1}{12}\left(k^3 + 2k^2t + 12kt - k + 2t^3 + 12t^2 - 16t + \frac{t^2-t^4}{k}\right)$$

with equality if and only if $G \cong U(k,t)$.

(ii) *For integers k,t with $k \geq 3$, $k \geq t \geq 1$, and $v \in V(G)$, we have*

$$Kf_G(v) \geq f(k,t)$$

with equality if and only if $G \cong U(k,t)$, and v is a central vertex of the t vertices of degree three in $U(k,t)$, where

$$f(k,t) = \begin{cases} \frac{1}{12}(2k^2 + 3t^2 + 12t - 5 - \frac{t^3-t}{k}) & \text{if } t \text{ is odd,} \\ \frac{1}{12}(2k^2 + 3t^2 + 12t - 2 - \frac{t^3+2t}{k}) & \text{if } t \text{ is even.} \end{cases}$$

Proof. First we prove (i). The cases $t = 1, k-2, k$ are trivial. Suppose that $t \neq 1, k-2, k$.

Let $S(G) = \{v \in V(C_k) : d_G(v) = 3\}$, and let $\sigma(G) = \sum_{\{v_i,v_j\}\subseteq S(G)} r_G(v_i, v_j)$. Clearly, $|S(G)| = t$.

If $t = 2$, say $S(G) = \{v_1, v_s\}$, then

$$\sigma(G) = r_G(v_1, v_s) \geq \frac{1 \cdot (k-1)}{k} = \sigma(U(k, 2))$$

with equality if and only if v_1 and v_s are adjacent in G, i.e., $G \cong U(k, 2)$.

If $t \geq 3$, then by Eq. (1), we have

$$\begin{aligned}
\sigma(U(k,t)) &= r_G(v_1, v_2) + r_G(v_1, v_3) + \cdots + r_G(v_1, v_t) \\
&\quad + r_G(v_2, v_3) + r_G(v_2, v_4) + \cdots + r_G(v_2, v_t) \\
&\quad + \cdots + r_G(v_{t-1}, v_t) \\
&= \sum_{i=1}^{t-1} \sum_{j=i+1}^{t} r_G(v_i, v_j) \\
&= \sum_{i=1}^{t-1} \sum_{j=i+1}^{t} \frac{(j-i) \cdot [k-(j-i)]}{k} \\
&= \sum_{i=1}^{t-1} \sum_{j=1}^{t-i} \frac{j \cdot (k-j)}{k} \\
&= \sum_{i=1}^{t-1} \sum_{j=1}^{i} \frac{j \cdot (k-j)}{k} \\
&= = \frac{1}{12k} t(t-1)(t+1)(2k-t).
\end{aligned} \tag{4}$$

Suppose that $t = 3$. Then k is odd as G has perfect matching. By symmetry, we may assume that $S(G) = \{v_1, v_i, v_j\}$ with $1 < i < j$, and $d_G(v_1, v_i) \leq d_G(v_1, v_j)$. Obviously, $i \leq \frac{k+1}{2}$. If $j \leq \frac{k+1}{2}$, then note that $d_G(v_1, v_j) \geq 2$, and by Eq. (1), we have

$$\begin{aligned}
\sigma(G) &= r_G(v_1, v_i) + r_G(v_1, v_j) + r_G(v_i, v_j) \\
&\geq \frac{1 \cdot (k-1)}{k} + \frac{2 \cdot (k-2)}{k} + \frac{1 \cdot (k-1)}{k} \\
&= \sigma(U(k, 3))
\end{aligned}$$

with equality if and only if $i = 2$ and $j = 3$, i.e., $G \cong U(k, 3)$. If $j = k$, then we have $i = 2$ since $d_G(v_1, v_i) \leq d_G(v_1, v_j)$, i.e., $G \cong U(k, 3)$. Note that $j \neq k-1$ as G has perfect matching. If $\frac{k+3}{2} \leq j \leq k-2$, then $d_G(v_1, v_j) \geq 3$, and by Eq. (1), we have

$$\begin{aligned}
\sigma(G) &= r_G(v_1, v_i) + r_G(v_1, v_j) + r_G(v_i, v_j) \\
&\geq \frac{1 \cdot (k-1)}{k} + \frac{3 \cdot (k-3)}{k} + \frac{1 \cdot (k-1)}{k}
\end{aligned}$$

$$
\begin{aligned}
&> \frac{1\cdot(k-1)}{k}+\frac{2\cdot(k-2)}{k}+\frac{1\cdot(k-1)}{k}\\
&= \sigma(U(k,3)).
\end{aligned}
$$

Now it follows that $\sigma(G)\geq\sigma(U(k,3))$ with equality if and only if $G\cong U(k,3)$.

Suppose that $t=k-4\geq 4$. Suppose to the contrary that $G\ncong U(k,k-4)$. Then there are two pairs of adjacent vertices of degree two on the cycle C_k in G, separated by $a\geq 1$ consecutive vertices $v_{i_1},v_{i_2},\ldots,v_{i_a}$ of degree three and $b\geq 1$ consecutive vertices $v_{j_1},v_{j_2},\ldots,v_{j_b}$ of degree three on the cycle C_k, where $d_{C_k}(v_{i_1},v_{j_1})=3$, $d_{C_k}(v_{i_a},v_{j_b})=3$, and $a+b=k-4$. Assume that $a\geq b$. Denote by w the pendent neighbor of v_{j_1} in G. Consider $G'=G-\{v_{j_1}w\}+\{vw\}$, where v is the neighbor of v_{i_1} with degree two on the cycle. Note that $S(G)=\{v_{i_1},v_{i_2},\ldots,v_{i_a},v_{j_1},v_{j_2},\ldots,v_{j_b}\}$ and $S(G')=\{v_{i_1},v_{i_2},\ldots,v_{i_a},v,v_{j_2},\ldots,v_{j_b}\}$. If $b\geq 2$, then by Eq. (1), we have

$$
\sum_{s=1}^{a} r_G(v_{j_1},v_{i_s})-\sum_{s=1}^{a} r_{G'}(v,v_{i_s})=\frac{1}{k}\sum_{i=3}^{a+2} i(k-i)-\frac{1}{k}\sum_{i=1}^{a} i(k-i),
$$

$$
\sum_{s=2}^{b} r_G(v_{j_1},v_{j_s})-\sum_{s=2}^{b} r_{G'}(v,v_{j_s})=\frac{1}{k}\sum_{i=1}^{b-1} i(k-i)-\frac{1}{k}\sum_{i=3}^{b+1} i(k-i),
$$

and thus

$$
\begin{aligned}
& \sigma(G)-\sigma(G')\\
=\ & \sum_{x\in S(G)\setminus\{v_{j_1}\}} r_G(v_{j_1},x)-\sum_{x\in S(G')\setminus\{v\}} r_{G'}(v,x)\\
=\ & \left(\sum_{s=1}^{a} r_G(v_{j_1},v_{i_s})+\sum_{s=2}^{b} r_G(v_{j_1},v_{j_s})\right)\\
& -\left(\sum_{s=1}^{a} r_{G'}(v,v_{i_s})+\sum_{s=2}^{b} r_{G'}(v,v_{j_s})\right)\\
=\ & \left(\sum_{s=1}^{a} r_G(v_{j_1},v_{i_s})-\sum_{s=1}^{a} r_{G'}(v,v_{i_s})\right)\\
& +\left(\sum_{s=2}^{b} r_G(v_{j_1},v_{j_s})-\sum_{s=2}^{b} r_{G'}(v,v_{j_s})\right)\\
=\ & \left(\frac{1}{k}\sum_{i=3}^{a+2} i(k-i)-\frac{1}{k}\sum_{i=1}^{a} i(k-i)\right)\\
& +\left(\frac{1}{k}\sum_{i=1}^{b-1} i(k-i)-\frac{1}{k}\sum_{i=3}^{b+1} i(k-i)\right)
\end{aligned}
$$

$$= \frac{4}{k}(a-b+1) > 0.$$

If $b = 1$, then by similar arguments as above, we have $\sigma(G) - \sigma(G') = \frac{4a}{k} > 0$. Thus $\sigma(G) > \sigma(G')$ for $b \geq 1$. By repeating the transformation from G to G', we may finally get $\sigma(G) > \sigma(U(k, k-4))$. Thus if $t = k - 4$, then $\sigma(G) \geq \sigma(U(k, k-4))$ with equality if and only if $G \cong U(k, k-4)$.

Suppose that $(k, t) = (10, 4)$. Then there are exactly four possibilities for G, and by suitable labeling, we may assume that $S(G) = \{v_1, v_2, v_3, v_4\}$, $\{v_1, v_2, v_3, v_6\}$, $\{v_1, v_2, v_5, v_6\}$, or $\{v_1, v_2, v_5, v_8\}$. By direct calculation, we have

$$\sigma(G) = \begin{cases} 8 & \text{if } S(G) = \{v_1, v_2, v_3, v_4\}, \\ \frac{52}{5} & \text{if } S(G) = \{v_1, v_2, v_3, v_6\}, \\ \frac{56}{5} & \text{if } S(G) = \{v_1, v_2, v_5, v_6\}, \\ 12 & \text{if } S(G) = \{v_1, v_2, v_5, v_8\}, \end{cases}$$

and thus $\sigma(G) \geq 8$ with equality if and only if $G \cong U(10, 4)$.

Suppose that $(k, t) = (11, 5)$. Then there are exactly five possibilities for G, and by suitable labeling, we may assume that $S(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $\{v_1, v_2, v_3, v_4, v_7\}$, $\{v_1, v_2, v_3, v_6, v_7\}$, $\{v_1, v_2, v_3, v_6, v_9\}$, or $\{v_1, v_2, v_5, v_8, v_9\}$. By direct calculation, we have

$$\sigma(G) = \begin{cases} \frac{170}{11} & \text{if } S(G) = \{v_1, v_2, v_3, v_4, v_5\}, \\ \frac{202}{11} & \text{if } S(G) = \{v_1, v_2, v_3, v_4, v_7\}, \\ \frac{218}{11} & \text{if } S(G) = \{v_1, v_2, v_3, v_6, v_7\}, \\ \frac{226}{11} & \text{if } S(G) = \{v_1, v_2, v_3, v_6, v_9\}, \\ \frac{234}{11} & \text{if } S(G) = \{v_1, v_2, v_5, v_8, v_9\}, \end{cases}$$

and thus $\sigma(G) \geq \frac{170}{11}$ with equality if and only if $G \cong U(11, 5)$.

Suppose that $(k, t) = (12, 4)$. Then there are exactly eight possibilities for G, and by suitable labeling, we may assume that $S(G) = \{v_1, v_2, v_3, v_4\}$, $\{v_1, v_2, v_3, v_6\}$, $\{v_1, v_2, v_3, v_8\}$, $\{v_1, v_2, v_5, v_6\}$, $\{v_1, v_2, v_7, v_8\}$, $\{v_1, v_2, v_5, v_8\}$, $\{v_1, v_2, v_5, v_{10}\}$, or $\{v_1, v_4, v_7, v_{10}\}$. By direct calculation, we have

$$\sigma(G) = \begin{cases} \frac{25}{3} & \text{if } S(G) = \{v_1, v_2, v_3, v_4\}, \\ \frac{34}{3} & \text{if } S(G) = \{v_1, v_2, v_3, v_6\}, \\ \frac{37}{3} & \text{if } S(G) = \{v_1, v_2, v_3, v_8\}, \\ \frac{37}{3} & \text{if } S(G) = \{v_1, v_2, v_5, v_6\}, \\ \frac{41}{3} & \text{if } S(G) = \{v_1, v_2, v_7, v_8\}, \\ 14 & \text{if } S(G) = \{v_1, v_2, v_5, v_8\}, \\ \frac{41}{3} & \text{if } S(G) = \{v_1, v_2, v_5, v_{10}\}, \\ 15 & \text{if } S(G) = \{v_1, v_4, v_7, v_{10}\}, \end{cases}$$

and thus $\sigma(G) \geq \frac{25}{3}$ with equality if and only if $G \cong U(12,4)$.

Combining all the above cases, and by Eq. (4), we can deduce that

$$\sigma(G) \geq \sigma(U(k,t)) = \frac{1}{12k} t(t-1)(t+1)(2k-t)$$

with equality if and only if $G \cong U(k,t)$ for $t = 1, 2, 3, k-4, k-2, k$, $(k,t) = (10,4)$, $(k,t) = (11,5)$, or $(k,t) = (12,4)$. For $1 \leq i \leq k$ with $d_G(v_i) = 3$, let u_i be the pendent neighbor of v_i in G.

By Eqs. (3) and (2), we have

$$\begin{aligned}
Kf(G) &= \sum_{\{v_i,v_j\}\subseteq V(C_k)} r_G(v_i,v_j) + \sum_{u_i\in V(G)\setminus V(C_k)} \sum_{v_j\in V(C_k)} r_G(u_i,v_j) \\
&\quad + \sum_{\{u_i,u_j\}\subseteq V(G)\setminus V(C_k)} r_G(u_i,u_j) \\
&= \frac{k^3-k}{12} + \sum_{v_i\in S(G)} \sum_{v_j\in V(C_k)} (1 + r_G(v_i,v_j)) + \sum_{\{v_i,v_j\}\subseteq S(G)} (2 + r_G(v_i,v_j)) \\
&= \frac{k^3-k}{12} + \sum_{v_i\in S(G)} (k + Kf_{C_k}(v_i)) + 2\binom{t}{2} + \sum_{\{v_i,v_j\}\subseteq S(G)} r_G(v_i,v_j) \\
&= \frac{k^3-k}{12} + t\left(k + \frac{k^2-1}{6}\right) + 2\binom{t}{2} + \sigma(G) \\
&\geq \frac{k^3-k}{12} + t\left(k + \frac{k^2-1}{6}\right) + 2\binom{t}{2} + \frac{1}{12k} t(t-1)(t+1)(2k-t) \\
&= \frac{1}{12}\left(k^3 + 2k^2t + 12kt - k + 2t^3 + 12t^2 - 16t + \frac{t^2-t^4}{k}\right)
\end{aligned}$$

with equality if and only if $G \cong U(k,t)$ for $t = 1, 2, 3, k-4, k-2, k$, $(k,t) = (10,4)$, $(k,t) = (11,5)$, or $(k,t) = (12,4)$.

Next we prove (ii). Let $v \in V(G)$. For $v_i \in V(C_k)$, clearly $Kf_G(v_i^*) - Kf_G(v_i) = 2m - 2 > 0$, where $d_G(v_i) = 3$, and v_i^* is the unique neighbor of v_i in G outside C_k. Thus we may assume that $v = v_i \in V(C_k)$. By Eq. (1), it is easily seen that

$$\begin{aligned}
\sum_{v_j\in S(G)} r_G(v_i,v_j) &\geq \begin{cases} 0 + 2\sum\limits_{i=1}^{(t-1)/2} \frac{i\cdot(k-i)}{k} & \text{if } t \text{ is odd} \\ 0 + 2\sum\limits_{i=1}^{(t-2)/2} \frac{i\cdot(k-i)}{k} + \frac{t/2\cdot(k-t/2)}{k} & \text{if } t \text{ is even} \end{cases} \\
&= \begin{cases} \frac{1}{12}(3t^2 - 3 - \frac{t^3-t}{k}) & \text{if } t \text{ is odd} \\ \frac{1}{12}(3t^2 - \frac{t^3+2t}{k}) & \text{if } t \text{ is even} \end{cases}
\end{aligned}$$

with equality if and only if the t vertices in $S(G)$ are consecutive on C_k, i.e., $G \cong U(k,t)$, and v_i is a central vertex of the t vertices of degree three in $U(k,t)$. For $v_i \in V(C_k)$, by

Eq. (2), we have

$$\begin{aligned}
Kf_G(v_i) &= Kf_{C_k}(v_i) + \sum_{u_j \in V(G)\setminus V(C_k)} r_G(v_i, u_j) \\
&= \frac{k^2-1}{6} + \sum_{v_j \in S(G)} (1 + r_G(v_i, v_j)) \\
&= \frac{k^2-1}{6} + t + \sum_{v_j \in S(G)} r_G(v_i, v_j) \\
&\geq \begin{cases} \frac{k^2-1}{6} + t + \frac{1}{12}(3t^2 - 3 - \frac{t^3-t}{k}) & \text{if } t \text{ is odd} \\ \frac{k^2-1}{6} + t + \frac{1}{12}(3t^2 - \frac{t^3+2t}{k}) & \text{if } t \text{ is even} \end{cases} \\
&= \begin{cases} \frac{1}{12}(2k^2 + 3t^2 + 12t - 5 - \frac{t^3-t}{k}) & \text{if } t \text{ is odd} \\ \frac{1}{12}(2k^2 + 3t^2 + 12t - 2 - \frac{t^3+2t}{k}) & \text{if } t \text{ is even} \end{cases} \\
&= f(k,t)
\end{aligned}$$

with equality if and only if $G \cong U(k,t)$, and v_i is a central vertex of the t vertices of degree three in $U(k,t)$. ■

If $G \in \mathbb{U}_1(m)$ with the unique cycle C_k and t pendent vertices, where $2 \leq m \leq 8$, then $t = 1, 2, 3, k-4, k-2, k$, $(k,t) = (10,4)$, $(k,t) = (11,5)$, or $(k,t) = (12,4)$. Now by Lemma 2.1 (i), we have

Lemma 2.2. *If G is a graph in $\mathbb{U}_1(m)$ with the minimum Kirchhoff index, where $2 \leq m \leq 8$, then $G \cong U(k,t)$ with $k+t = 2m$, $k \geq 3$ and $k \geq t \geq 1$.*

2.2 The Kirchhoff index of graphs in $\mathbb{U}_2(m)$ with small m

The following result will be useful for comparing the Kirchhoff indices of graphs.

For simplicity, let $|G| = |V(G)|$ for a graph G.

Lemma 2.3. [12] *Let G and H be two connected graphs with $u \in V(G)$ and $w \in V(H)$. Let GuH be the graph obtained from G and H by identifying $u \in V(G)$ with $w \in V(H)$. Then*

$$Kf(GuH) = Kf(G) + Kf(H) + (|H|-1)\,Kf_G(u) + (|G|-1)\,Kf_H(w).$$

Let P_n be the path on n vertices.

If u is a pendent vertex being adjacent to a vertex v of degree two in the graph G, then the path of G induced by the vertices u and v is said to be a pendent P_2 of G. Clearly, every graph in $\mathbb{U}_2(m)$ has at least one pendent P_2.

For a given graph $G \in \mathbb{U}_2(m)$, starting from G, deleting the pendent P_2's repeatedly, until there is no pendent P_2, the resulting graph is denoted by $\bar{G}$. Let $\bar{n} = |\bar{G}|$. Clearly, $\bar{G} \in \mathbb{U}_1(\frac{\bar{n}}{2}) \cup \{C_{\bar{n}}\}$.

Now we determine the minimum Kirchhoff index among the graphs in $\mathbb{U}_2(m)$ with $3 \leq m \leq 8$.

Lemma 2.4. *If G is a graph in $\mathbb{U}_2(m)$ with the minimum Kirchhoff index, where $3 \leq m \leq 8$, then $G \cong U(k,t,0,j)$ with $k+t+2j = 2m$, $k \geq 3$, $k \geq t \geq 1$ and $j \geq 1$.*

Proof. Let $G \in \mathbb{U}_2(m)$, and k be the length of the unique cycle of G.

Denote by the deleting process from G to $\bar{G}$ as follows:

$$G = G_1 \rightarrow G_2 \rightarrow \cdots \rightarrow G_{r-1} \rightarrow G_r = \bar{G},$$

where G_{i+1} is the (unicyclic) graph obtained from G_i by deleting a pendent P_2, where $1 \leq i \leq r-1$. Note that $\bar{n} + 2(r-1) = 2m$.

Recall that $G_r = \bar{G} \in \mathbb{U}_1(\frac{\bar{n}}{2}) \cup \{C_{\bar{n}}\}$. By Lemma 2.2, we have $Kf(G_r) \geq Kf(U(k,t,0,0))$, where $k+t = \bar{n}$. Moreover, by Lemma 2.1 (ii) and Lemma 2.3, we have $Kf(G_{r-1}) \geq Kf(U(k,t,0,1))$ with equality if and only if $G_{r-1} \cong U(k,t,0,1)$. Again by Lemma 2.1 (ii) and Lemma 2.3, we have $Kf(G_{r-2}) \geq Kf(U(k,t,0,2))$ with equality if and only if $G_{r-2} \cong U(k,t,0,2)$. Repeating the arguments, finally we can deduce that $Kf(G) = Kf(G_1) \geq Kf(U(k,t,0,r-1))$ with last equality if and only if $G \cong U(k,t,0,r-1)$.

Then the result follows easily. ■

2.3 The effect on the Kirchhoff index of graphs under the deletion of some vertices

First we introduce a unicyclic graph.

Let $U_{n,m} = U(5,1,n-2m,m-3)$, where $3 \leq m \leq \lfloor \frac{n}{2} \rfloor$, see Fig 2. It is easily checked that

$$Kf(U_{n,m}) = n^2 + nm - 5n - 3m + 4. \tag{5}$$

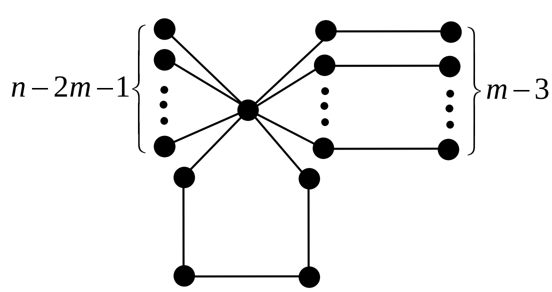

Figure 2: The graph $U_{n,m}$.

Next we establish a lower bound of $Kf_G(u)$, where $G \in \mathbb{U}(n,m)$ and $u \in V(G)$.

Lemma 2.5. *Let $G \in \mathbb{U}(n,m)$ with the unique cycle C_k, where $n \geq 6$, $m \geq 3$, $k \geq 3$. If $T_i \cong P_1$ or P_2 for $2 \leq i \leq k$, then for $u \in V(T_1)$,*

$$Kf_G(u) \geq n + m - 4$$

with equality if and only if $G \cong U_{n,m}$, and u is the vertex of maximum degree in $U_{n,m}$.

Proof. Let M be a maximum matching of G. First we establish an upper bound of $d_G(u)$. Let

$$A_1 = \{xy \in E(G) \setminus M : \text{either } x = u \text{ or } y = u\},$$
$$A_2 = \{xy \in E(G) \setminus M : x, y \neq u \text{ and } xy \in E(C_k)\},$$
$$A_3 = \{xy \in E(G) \setminus M : x, y \neq u \text{ and } xy \notin E(C_k)\}.$$

Clearly, A_1, A_2, A_3 are pairwise disjoint, and $E(G) \setminus M = A_1 \cup A_2 \cup A_3$. Thus

$$|E(G) \setminus M| = n - m = |A_1| + |A_2| + |A_3|. \tag{6}$$

Note that

(a) $|A_1| \geq d_G(u) - 1$ with equality if and only if u is M-saturated;

(b) $|A_2| \geq \lfloor \frac{k-2}{2} \rfloor$ if u lies on the unique cycle C_k of G, and $|A_2| \geq \lfloor \frac{k+1}{2} \rfloor$ if u lies outside the unique cycle C_k of G;

(c) $|A_3| \geq 0$.

It follows from Eq. (6) that if u lies on the unique cycle C_k of G, then $n - m \geq (d_G(u) - 1) + \lfloor \frac{k-2}{2} \rfloor$, i.e.,

$$d_G(u) \leq n - m + 1 - \left\lfloor \frac{k-2}{2} \right\rfloor \tag{7}$$

with equality if and only if the corresponding equalities in (a), (b), (c) hold, while if u lies outside the unique cycle C_k of G, then $n - m \geq (d_G(u) - 1) + \lfloor \frac{k+1}{2} \rfloor$, i.e.,

$$d_G(u) \leq n - m + 1 - \left\lfloor \frac{k+1}{2} \right\rfloor \tag{8}$$

with equality if and only if the corresponding equalities in (a), (b), (c) hold.

Case 1. u lies on the unique cycle C_k of G.

Subcase 1.1. k is odd and $T_i \cong P_1$ for $2 \leq i \leq k$.

By Eq. (2) and inequality (7), we have

$$Kf_G(u) \;=\; Kf_{C_k}(u) + \sum_{x \in V(G) \setminus V(C_k)} r_G(u, x)$$

$$
\begin{aligned}
&\geq \frac{k^2-1}{6} + [(d_G(u)-2) + 2(n-k-d_G(u)+2)] \\
&= -d_G(u) + \frac{1}{6}k^2 - 2k + 2n + \frac{11}{6} \\
&\geq -\left(n-m+1-\frac{k-3}{2}\right) + \frac{1}{6}k^2 - 2k + 2n + \frac{11}{6} \\
&= \frac{1}{6}(k^2 - 9k + 6n + 6m - 4)
\end{aligned}
$$

with equality if and only if $G \cong U(k, 1, n-2m, m-\frac{k+1}{2})$ with odd k, and u is the vertex of maximum degree in $U(k, 1, n-2m, m-\frac{k+1}{2})$.

Subcase 1.2. k is even and $T_i \cong P_1$ for $2 \leq i \leq k$, or there is at least one of T_i such that $T_i \cong P_2$ for $2 \leq i \leq k$.

Obviously, $r_G(u, v) \geq 1 + \frac{2\cdot(k-2)}{k} \geq 2$, where v is the unique pendent neighbor of v_i with $3 \leq i \leq k-1$ if $T_i \cong P_2$. On the other hand, we also note that if k is odd, then $T_i \cong P_2$ for some i with $2 \leq i \leq k$, and thus either $|A_2| > \lfloor \frac{k-2}{2} \rfloor = \frac{k-3}{2}$ or $|A_3| > 0$, by Eq. (6), we have $n - m \geq (d_G(u) - 1) + \frac{k-3}{2} + 1$, i.e.,

$$
d_G(u) \leq n - m - \frac{k-3}{2}. \tag{9}
$$

Let a be the number of pendent vertices attached to v_2 or v_k in G, where $0 \leq a \leq 2$. Then by Eq. (2), and inequalities (7) and (9), we have

$$
\begin{aligned}
Kf_G(u) &= Kf_{C_k}(u) + \sum_{x \in V(G)\setminus V(C_k)} r_G(u, x) \\
&\geq \frac{k^2-1}{6} + \left[(d_G(u)-2) + \left(1 + \frac{1\cdot(k-1)}{k}\right)a + 2(n-k-a-d_G(u)+2)\right] \\
&= -\frac{a}{k} - d_G(u) + \frac{1}{6}k^2 - 2k + 2n + \frac{11}{6} \\
&\geq \begin{cases} -\frac{2}{k} - \left(n-m+1-\frac{k-2}{2}\right) + \frac{1}{6}k^2 - 2k + 2n + \frac{11}{6} & \text{if } k \text{ is even} \\ -\frac{2}{k} - \left(n-m-\frac{k-3}{2}\right) + \frac{1}{6}k^2 - 2k + 2n + \frac{11}{6} & \text{if } k \text{ is odd} \end{cases} \\
&= \begin{cases} \frac{1}{6}(k^2 - 9k + 6n + 6m - 1 - \frac{12}{k}) & \text{if } k \text{ is even} \\ \frac{1}{6}(k^2 - 9k + 6n + 6m + 2 - \frac{12}{k}) & \text{if } k \text{ is odd} \end{cases} \\
&\geq \frac{1}{6}(k^2 - 9k + 6n + 6m - 4).
\end{aligned}
$$

If $Kf_G(u) = \frac{1}{6}(k^2-9k+6n+6m-4)$, then $a = 2$, $k = 4$ and $d_G(u) = n-m+1-\frac{k-2}{2} = n-m$. However, $a = 2$ and $k = 4$ imply that either $|A_2| > 1$ or $|A_3| > 0$, and thus by Eq. (6), we have $n - m > d_G(u)$. Therefore $Kf_G(u) > \frac{1}{6}(k^2 - 9k + 6n + 6m - 4)$.

Case 2. u lies outside the unique cycle C_k of G.

Note that for $v \notin V(C_k) \cup V(T_1)$, $r_G(u,v) \geq 2 + \frac{1\cdot(k-1)}{k} > 2$. Let b be the number of neighbors of u on C_k, where $b = 0, 1$. Now by Eq. (2) and inequality (8), we have

$$\begin{aligned}
Kf_G(u) &= Kf_{C_k}(u) + \sum_{x \in V(G)\setminus V(C_k)} r_G(u,x) \\
&\geq \left(k + \frac{k^2-1}{6}\right) + [(d_G(u) - b) + 2(n - k - d_G(u) + b)] \\
&= b - d_G(u) + \frac{1}{6}k^2 - k + 2n - \frac{1}{6} \\
&\geq 0 - \left(n - m + 1 - \left\lfloor \frac{k+1}{2} \right\rfloor\right) + \frac{1}{6}k^2 - k + 2n - \frac{1}{6} \\
&> \frac{1}{6}(k^2 - 9k + 6n + 6m - 4).
\end{aligned}$$

Now combining Cases 1 and 2, we have

$$Kf_G(u) \geq \frac{1}{6}(k^2 - 9k + 6n + 6m - 4)$$

with equality if and only if $G \cong U(k, 1, n-2m, m - \frac{k+1}{2})$ with odd k, and u is the vertex of maximum degree in $U(k, 1, n-2m, m - \frac{k+1}{2})$. Thus

$$\begin{aligned}
Kf_G(u) \geq \frac{1}{6}(k^2 - 9k + 6n + 6m - 4) &\geq \frac{1}{6}(5^2 - 9\cdot 5 + 6n + 6m - 4) \\
&= n + m - 4
\end{aligned}$$

with equalities if and only if $G \cong U(5, 1, n-2m, m-3) = U_{n,m}$, and u is the vertex of maximum degree in $U_{n,m}$. ■

Now we present a stronger version of lemma 2.5.

Lemma 2.6. *Let G be a unicyclic graph with n vertices and matching number at least m, where $n \geq 6$, $m \geq 3$. For $u \in V(G)$,*

$$Kf_G(u) \geq n + m - 4$$

with equality if and only if $G \cong U_{n,m}$, and u is the vertex of maximum degree in $U_{n,m}$.

Proof. Let G be a unicyclic graph with a vertex $u \in V(G)$ such that

$$Kf_G(u) = \min\{Kf_H(x) : H \in \mathbb{U}(n,r), x \in V(H), r \geq m\}. \tag{10}$$

Assume that $u \in V(T_1)$. Let M be a maximum matching of G. Suppose that $|T_i| \geq 3$ for some i with $2 \leq i \leq k$, where k is the length of the unique cycle of G. Then there is some edge, say xy, in T_i outside M. Assume that the vertices x and u lie in the same component of $G - xy$. Let $G_1 = G - xy + uy$. Clearly, M is also a matching of G_1,

and thus G_1 has matching number at least m. However, $Kf_{G_1}(u) < Kf_G(u)$, which is a contradiction. Thus $|T_i| = 1, 2$, i.e., $T_i \cong P_1$ or P_2 for $2 \le i \le k$. By Lemma 2.5, we have

$$Kf_G(u) \ge n + r - 4 \ge n + m - 4$$

with equalities if and only if $G \cong U_{n,m}$, and u is the vertex of maximum degree in $U_{n,m}$. ■

The following result turns out to be of rather important for the proof of our main results.

Lemma 2.7. *Let $G \in \mathbb{U}(n, m)$ with a pendent vertex x being adjacent to vertex y, and let z be the neighbor of y different from x if $d_G(y) = 2$, where $n \ge 6$, $m \ge 3$. Then*

$$Kf(G) - Kf(G - x) \ge 2n + m - 6$$

with equality if and only if $G \cong U_{n,m}$, and x is a pendent neighbor of the vertex of maximum degree in $U_{n,m}$. Moreover, if $d_G(y) = 2$, then

$$Kf(G) - Kf(G - x - y) \ge 5n + 2m - 19$$

with equality if and only if $G \cong U_{n,m}$.

Proof. Note that $Kf_G(x) - Kf_G(y) = n - 2$. Then by Lemma 2.6, we have

$$\begin{aligned} Kf(G) - Kf(G - x) &= Kf_G(x) \\ &= Kf_G(y) + n - 2 \\ &\ge (n + m - 4) + n - 2 = 2n + m - 6 \end{aligned}$$

with equality if and only if $G \cong U_{n,m}$, and x is a pendent neighbor of the vertex of maximum degree in $U_{n,m}$.

If $d_G(y) = 2$, then $Kf_G(y) - Kf_G(z) = n - 4$, and thus by Lemma 2.6, we have

$$\begin{aligned} Kf(G) - Kf(G - x - y) &= Kf_G(x) + Kf_G(y) - 1 \\ &= 2Kf_G(z) + 3n - 11 \\ &\ge 2(n + m - 4) + 3n - 11 = 5n + 2m - 19 \end{aligned}$$

with equality if and only if $G \cong U_{n,m}$. ■

Lemma 2.8. [11] *Let G be an n-vertex unicyclic graph with the unique cycle C_k, where $3 \le k \le n - 1$. Then*

$$Kf(G) \ge \frac{1}{12}[-k^3 + 2nk^2 - (12n - 13)k + 12n^2 - 14n]$$

with equality if and only if $G \cong U(k, 1, n - k - 1, 0)$.

3 Results

First we consider the minimum Kirchhoff index of unicyclic graphs with perfect matching.

Theorem 3.1. *Among the graphs in* $\mathbb{U}(2m, m)$ *with* $m \geq 2$, C_{2m} *for* $2 \leq m \leq 4$, $U(8, 2)$ *for* $m = 5$, $U(8, 4)$ *for* $m = 6$, $U(7, 7)$ *for* $m = 7$, *and* $U_{2m,m}$ *for* $m \geq 8$ *are the unique graphs with the minimum Kirchhoff indices, which are equal to* $\frac{1}{6}(4m^3 - m)$ *for* $2 \leq m \leq 4$, $81\frac{7}{8}$ *for* $m = 5$, $135\frac{1}{2}$ *for* $m = 6$, 203 *for* $m = 7$, *and* $6m^2 - 13m + 4$ *for* $m \geq 8$.

Proof. Recall that $\mathbb{U}(2m, m) = \mathbb{U}_1(m) \cup \mathbb{U}_2(m) \cup \{C_{2m}\}$.
Case 1. $2 \leq m \leq 8$.

The case $m = 2$ is obvious since $\mathbb{U}(4, 2) = \{U(3, 1), C_4\}$, where

$$Kf(U(3, 1)) = 6\frac{1}{3} > 5 = Kf(C_4).$$

For $3 \leq m \leq 8$, by Lemmas 2.2 and 2.4, the minimum Kirchhoff index of the graphs in $\mathbb{U}(2m, m)$ is precisely achieved by some graph of the form $U(k, t, 0, j)$, where $k + t + 2j = 2m$, $k \geq 3$, $k \geq t \geq 0$ and $j \geq 0$. In Tables 1–6 corresponding to $m = 3, 4, \ldots, 8$, we list these graphs and their Kirchhoff indices. We use $(k, t; j)$ to represent the graph $U(k, t, 0, j)$ in these tables. From these tables, we find that

(1) $U(6, 0, 0, 0) = C_6$ is the unique graph in $\mathbb{U}(6, 3)$ with the minimum Kirchhoff index, which is equal to $17\frac{1}{2}$;

(2) $U(8, 0, 0, 0) = C_8$ is the unique graph in $\mathbb{U}(8, 4)$ with the minimum Kirchhoff index, which is equal to 42;

(3) $U(8, 2, 0, 0) = U(8, 2)$ is the unique graph in $\mathbb{U}(10, 5)$ with the minimum Kirchhoff index, which is equal to $81\frac{7}{8}$;

(4) $U(8, 4, 0, 0) = U(8, 4)$ is the unique graph in $\mathbb{U}(12, 6)$ with the minimum Kirchhoff index, which is equal to $135\frac{1}{2}$;

(5) $U(7, 7, 0, 0) = U(7, 7)$ is the unique graph in $\mathbb{U}(14, 7)$ with the minimum Kirchhoff index, which is equal to 203;

(6) $U(5, 1, 0, 5) = U_{16,8}$ is the unique graph in $\mathbb{U}(16, 8)$ with the minimum Kirchhoff index, which is equal to 284.

Case 2. $m \geq 9$.

We prove the result by induction on m. Suppose that the result holds for all the graphs in $\mathbb{U}(2m - 2, m - 1)$. Let $G \in \mathbb{U}(2m, m)$.

If $G \cong C_{2m}$, then by Eqs. (3) and (5), we have

$$Kf(G) = \frac{1}{6}(4m^3 - m) > 6m^2 - 13m + 4 = Kf(U_{2m,m}).$$

Table 1: The Kirchhoff indices of the graphs $U(k, t, 0, j)$ in $\mathbb{U}(6, 3)$.

Graphs	$(3, 1; 1)$	$(3, 3; 0)$	$(4, 0; 1)$	$(4, 2; 0)$	$(5, 1; 0)$	$(6, 0; 0)$
Kirchhoff indices	24	23	23	$20\frac{3}{4}$	19	$17\frac{1}{2}$

Table 2: The Kirchhoff indices of the graphs $U(k, t, 0, j)$ in $\mathbb{U}(8, 4)$.

Graphs	$(3, 1; 2)$	$(3, 3; 1)$	$(4, 0; 2)$	$(4, 2; 1)$	$(4, 4; 0)$	$(5, 1; 1)$
Kirchhoff indices	$53\frac{2}{3}$	$53\frac{1}{3}$	53	$50\frac{1}{4}$	48	48
Graphs	$(5, 3; 0)$	$(6, 0; 1)$	$(6, 2; 0)$	$(7, 1; 0)$	$(8, 0; 0)$	
Kirchhoff indices	$45\frac{4}{5}$	$48\frac{1}{6}$	44	43	42	

Table 3: The Kirchhoff indices of the graphs $U(k, t, 0, j)$ in $\mathbb{U}(10, 5)$.

Graphs	$(3, 1; 3)$	$(3, 3; 2)$	$(4, 0; 3)$	$(4, 2; 2)$	$(4, 4; 1)$	$(5, 1; 2)$
Kirchhoff indices	$95\frac{1}{3}$	$95\frac{2}{3}$	95	$91\frac{3}{4}$	91	89
Graphs	$(5, 3; 1)$	$(5, 5; 0)$	$(6, 0; 2)$	$(6, 2; 1)$	$(6, 4; 0)$	$(7, 1; 1)$
Kirchhoff indices	88	85	$90\frac{5}{6}$	$86\frac{1}{3}$	$83\frac{1}{2}$	86
Graphs	$(7, 3; 0)$	$(8, 0; 1)$	$(8, 2; 0)$	$(9, 1; 0)$	$(10, 0; 0)$	
Kirchhoff indices	$82\frac{1}{7}$	88	$81\frac{7}{8}$	$82\frac{1}{3}$	$82\frac{1}{2}$	

Suppose that $G \in \mathbb{U}_1(m)$. Recall that G is a graph of maximum degree three obtainable by attaching some pendent vertices to a cycle C_k, where $m \leq k \leq 2m - 1$. If $k = m, m + 1, m + 2$, then there are, respectively, $m, m - 1, m - 2$ pendent vertices in G outside the cycle C_k, and thus by Lemma 2.1 (i), we have

$$\begin{aligned} Kf(G) &\geq \begin{cases} \frac{1}{3}(m^3 + 6m^2 - 4m) & \text{if } k = m \\ \frac{1}{3}\left(m^3 + 7m^2 - 11m + 6 - \frac{3}{m+1}\right) & \text{if } k = m + 1 \\ \frac{1}{3}\left(m^3 + 8m^2 - 20m + 30 - \frac{60}{m+2}\right) & \text{if } k = m + 2 \end{cases} \\ &> 6m^2 - 13m + 4 = Kf(U_{2m,m}). \end{aligned}$$

If $m + 3 \leq k \leq 2m - 1$, then by Lemma 2.8, $Kf(G) \geq \frac{1}{12}h(k)$, where $h(k) = -k^3 + 4mk^2 - (24m - 13)k + 48m^2 - 28m$. Clearly, $h'(k) = -3k^2 + 8mk - 24m + 13$. Note that $h'(m+3) = 5m^2 - 18m - 14 > 0$ and $h'(2m-1) = 4m^2 - 20m + 10 > 0$. This implies that $h'(k) > 0$ for $m + 3 \leq k \leq 2m - 1$, i.e., $h(k)$ is increasing for k with $m + 3 \leq k \leq 2m - 1$.

Table 4: The Kirchhoff indices of the graphs $U(k,t,0,j)$ in $\mathbb{U}(12,6)$.

Graphs	$(3,1;4)$	$(3,3;3)$	$(4,0;4)$	$(4,2;3)$	$(4,4;2)$	$(5,1;3)$
Kirchhoff indices	149	150	149	$145\frac{1}{4}$	146	142
Graphs	$(5,3;2)$	$(5,5;1)$	$(6,0;3)$	$(6,2;2)$	$(6,4;1)$	$(6,6;0)$
Kirchhoff indices	$142\frac{1}{5}$	142	$145\frac{1}{2}$	$140\frac{2}{3}$	$140\frac{1}{6}$	136
Graphs	$(7,1;2)$	$(7,3;1)$	$(7,5;0)$	$(8,0;2)$	$(8,2;1)$	$(8,4;0)$
Kirchhoff indices	141	$138\frac{4}{7}$	$135\frac{6}{7}$	146	$139\frac{5}{8}$	$135\frac{1}{2}$
Graphs	$(9,1;1)$	$(9,3;0)$	$(10,0;1)$	$(10,2;0)$	$(11,1;0)$	$(12,0;0)$
Kirchhoff indices	142	$136\frac{1}{3}$	$146\frac{1}{2}$	$138\frac{2}{5}$	141	143

Table 5: The Kirchhoff indices of the graphs $U(k,t,0,j)$ in $\mathbb{U}(14,7)$.

Graphs	$(3,1;5)$	$(3,3;4)$	$(4,0;5)$	$(4,2;4)$	$(4,4;3)$	$(5,1;4)$
Kirchhoff indices	$214\frac{2}{3}$	$216\frac{1}{3}$	215	$210\frac{3}{4}$	213	207
Graphs	$(5,3;3)$	$(5,5;2)$	$(6,0;4)$	$(6,2;3)$	$(6,4;2)$	$(6,6;1)$
Kirchhoff indices	$208\frac{2}{5}$	211	$212\frac{1}{6}$	207	$208\frac{5}{6}$	$208\frac{1}{3}$
Graphs	$(7,1;3)$	$(7,3;2)$	$(7,5;1)$	$(7,7;0)$	$(8,0;3)$	$(8,2;2)$
Kirchhoff indices	208	207	208	203	216	$209\frac{3}{8}$
Graphs	$(8,4;1)$	$(8,6;0)$	$(9,1;2)$	$(9,3;1)$	$(9,5;0)$	$(10,0;2)$
Kirchhoff indices	208	$204\frac{7}{8}$	$213\frac{2}{3}$	$209\frac{5}{9}$	$206\frac{1}{9}$	$222\frac{1}{2}$
Graphs	$(10,2;1)$	$(10,4;0)$	$(11,1;1)$	$(11,3;0)$	$(12,0;1)$	$(12,2;0)$
Kirchhoff indices	$214\frac{1}{5}$	$208\frac{1}{2}$	220	$212\frac{9}{20}$	$227\frac{2}{3}$	$217\frac{7}{12}$
Graphs	$(13,1;0)$	$(14,0;0)$				
Kirchhoff indices	223	$227\frac{1}{2}$				

Thus

$$Kf(G) \geq \frac{1}{12}h(m+3) = \frac{1}{4}(m^3 + 13m^2 - 26m + 4) > 6m^2 - 13m + 4 = Kf(U_{2m,m}).$$

Now suppose that $G \in \mathbb{U}_2(m)$. Denote by x a pendent vertex in G whose unique neighbor y is of degree two, and z the neighbor of y different from x in G. Obviously, $xy \in M$. Then $G - x - y \in \mathbb{U}(2m-2, m-1)$, and thus by Lemma 2.7 and the induction

Table 6: The Kirchhoff indices of the graphs $U(k,t,0,j)$ in $\mathbb{U}(16,8)$.

Graphs	$(3,1;6)$	$(3,3;5)$	$(4,0;6)$	$(4,2;5)$	$(4,4;4)$	$(5,1;5)$
Kirchhoff indices	$292\frac{1}{3}$	$293\frac{2}{3}$	293	$288\frac{1}{4}$	292	284
Graphs	$(5,3;4)$	$(5,5;3)$	$(6,0;5)$	$(6,2;4)$	$(6,4;3)$	$(6,6;2)$
Kirchhoff indices	$286\frac{3}{5}$	292	$290\frac{5}{6}$	$285\frac{1}{3}$	$289\frac{1}{2}$	$292\frac{2}{3}$
Graphs	$(7,1;4)$	$(7,3;3)$	$(7,5;2)$	$(7,7;1)$	$(8,0;4)$	$(8,2;3)$
Kirchhoff indices	287	$287\frac{3}{7}$	$292\frac{1}{7}$	292	298	$291\frac{1}{8}$
Graphs	$(8,4;2)$	$(8,6;1)$	$(8,8;0)$	$(9,1;3)$	$(9,3;2)$	$(9,5;1)$
Kirchhoff indices	$292\frac{1}{2}$	$294\frac{1}{8}$	288	$297\frac{1}{3}$	$294\frac{7}{9}$	$295\frac{5}{9}$
Graphs	$(9,7;0)$	$(10,0;3)$	$(10,2;2)$	$(10,4;1)$	$(10,6;0)$	$(11,1;2)$
Kirchhoff indices	$292\frac{5}{9}$	$310\frac{1}{2}$	302	$299\frac{3}{10}$	296	311
Graphs	$(11,3;1)$	$(11,5;0)$	$(12,0;2)$	$(12,2;1)$	$(12,4;0)$	$(13,1;1)$
Kirchhoff indices	$305\frac{1}{11}$	$300\frac{5}{11}$	$324\frac{1}{3}$	$314\frac{1}{12}$	$306\frac{2}{3}$	324
Graphs	$(13,3;0)$	$(14,0;1)$	$(14,2;0)$	$(15,1;0)$	$(16,0;0)$	
Kirchhoff indices	$314\frac{7}{13}$	$335\frac{1}{2}$	$323\frac{3}{7}$	$332\frac{1}{3}$	340	

hypothesis, we have

$$\begin{aligned} Kf(G) &\geq Kf(G-x-y)+12m-19 \\ &\geq Kf(U_{2m-2,m-1})+12m-19 \\ &= 6m^2-13m+4 = Kf(U_{2m,m}) \end{aligned}$$

with equalities if and only if $G \cong U_{2m,m}$.

Then the result for $m \geq 9$ follows easily. ■

The remainder of the paper will focus on the minimum Kirchhoff index among the graphs in $\mathbb{U}(n,m)$, where $n > 2m$ and $m \geq 3$.

Lemma 3.1. [2] *Let $G \in \mathbb{U}(n,m) \setminus \{C_n\}$, where $n > 2m$, $m \geq 3$. Then there is a maximum matching M and a pendent vertex u of G such that u is not M-saturated.*

For a given graph $G \in \mathbb{U}(n,m) \setminus \{C_n\}$, where $n > 2m$, $m \geq 3$, by Lemma 3.1, there is a maximum matching M and a pendent vertex which is not M-saturated, after deleting this pendent vertex, we can get a graph in $\mathbb{U}(n-1,m)$. Repeating the process until it is exhausted, the resulting graph is denoted by G_0. Note that $G_0 \in \mathbb{U}(2m,m)$. Let $n_0 = |G_0|$. Furthermore, for the vertex $u \in V(G_0)$ satisfying $Kf_{G_0}(u)$ is minimum, let G_0^* be the graph obtained from G_0 by attaching $n-n_0$ pendent vertices to u.

Lemma 3.2. *Let $G \in \mathbb{U}(n,m) \setminus \{C_n\}$, where $n > 2m$, $m \geq 3$.*

(*i*) *Then $Kf(G) \geq Kf(G_0^*)$. In particular, if u is the unique vertex in G_0 such that $Kf_{G_0}(u)$ is minimum, then $Kf(G) \geq Kf(G_0^*)$ with equality if and only if $G \cong G_0^*$.*

(*ii*) *If $G_0 \ncong U_{n_0,m}$ and $Kf(G_0) \geq Kf(U_{n_0,m})$, then $Kf(G) > Kf(U_{n,m})$.*

Proof. Similar to the proof of Lemma 2.4, and by Lemma 2.3 repeatedly, (i) follows easily. On the other hand, by Lemmas 2.3 and 2.6, $Kf(G_0^*) > Kf(U_{n,m})$ follows from the hypothesis that $G_0 \ncong U_{n_0,m}$ and $Kf(G_0) \geq Kf(U_{n_0,m})$. Now together with $Kf(G) \geq Kf(G_0^*)$, we can get $Kf(G) > Kf(U_{n,m})$. ■

The following lemma reveal the possible graph with the minimum Kirchhoff index among the graphs in $\mathbb{U}(n,m) \setminus \{C_n\}$, where $n > 2m$ and $3 \leq m \leq 7$.

Lemma 3.3. *If G is a graph in $\mathbb{U}(n,m) \setminus \{C_n\}$ with the minimum Kirchhoff index, where $n > 2m$ and $3 \leq m \leq 7$, then $G_0 \cong U(k,t,0,j)$ with $k+t+2j = n_0$, $k \geq 3$, $k \geq t \geq 0$ and $j \geq 0$.*

Proof. Let $G \in \mathbb{U}(n,m) \setminus \{C_n\}$. Denote by k the length of the unique cycle of G. Suppose that there are t pendent vertices of G whose unique neighbors are all on the unique cycle of G. Note that $G_0 \in \mathbb{U}(2m,m)$, i.e., $G_0 \in \mathbb{U}_1(m) \cup \mathbb{U}_2(m)$.

Case 1. $G_0 \in \mathbb{U}_1(m)$.

First, by Lemma 3.2 (i), we have $Kf(G) \geq Kf(G_0^*)$. Next, by Lemma 2.2, we have $Kf(G_0) \geq Kf(U(k,t))$ with equality if and only if $G_0 \cong U(k,t)$), and thus by Lemma 2.1 (ii) and Lemma 2.3, we have $Kf(G_0^*) \geq Kf(U(k,t,i,0))$ with equality if and only if $G_0^* \cong U(k,t,i,0)$), where $k+t+i = n$ and $i \geq 1$. Now it follows that $Kf(G) \geq Kf(U(k,t,i,0))$ with equality if and only if $G \cong U(k,t,i,0)$.

Case 2. $G_0 \in \mathbb{U}_2(m)$.

Recall that, starting from G_0, deleting the pendent P_2's repeatedly, until there is no pendent P_2, the resulting graph is denoted by $\bar{G_0}$. Let $\bar{n_0} = |\bar{G_0}|$. Clearly, $\bar{G_0} \in \mathbb{U}_1(\frac{\bar{n_0}}{2}) \cup \{C_{\bar{n_0}}\}$.

Suppose that u is a vertex in $\bar{G_0}$ satisfying $Kf_{\bar{G_0}}(u)$ is minimum, and let H be the graph obtained from $\bar{G_0}$ by attaching i pendent vertices and j paths on two vertices to u.

Similar to the proof of Lemma 2.4, and by Lemma 2.3 repeatedly, $Kf(G) \geq Kf(H)$ follows easily.

On the other hand, recall that $\bar{G_0} \in \mathbb{U}_1(\frac{\bar{n_0}}{2}) \cup \{C_{\bar{n_0}}\}$, by Lemma 2.2, we have $Kf(\bar{G_0}) \geq Kf(U(k,t))$ with equality if and only if $\bar{G_0} \cong U(k,t)$), where $k+t = \bar{n_0}$, and thus by Lemma 2.1 (ii) and Lemma 2.3, we have $Kf(H) \geq Kf(U(k,t,i,j))$ with equality if and only if $H \cong U(k,t,i,j)$, where $k+t+i+2j = n$.

Now it follows that $Kf(G) \geq Kf(U(k,t,i,j))$ with equality if and only if $G \cong U(k,t,i,j)$.

Combining Cases 1 and 2, we have $Kf(G) \geq Kf(U(k,t,i,j))$ with equality if and only if $G \cong U(k,t,i,j)$, and $G \cong U(k,t,i,j)$ implies that $G_0 \cong U(k,t,0,j)$. ■

Now we determine the minimum Kirchhoff index among the unicyclic graphs with given matching number.

Theorem 3.2. *Among the graphs in $\mathbb{U}(n,m)$ with $2 \le m \le \lfloor \frac{n}{2} \rfloor$,*

(i) for $m = 2$, C_n for $n = 4,5$, $U(4,1,n-5,0)$ for $6 \le n \le 11$, $U(3,1,8,0)$ and $U(4,1,7,0)$ for $n = 12$, and $U(3,1,n-4,0)$ for $n \ge 13$ are the unique graphs with the minimum Kirchhoff indices, which are equal to $\frac{n^3-n}{12}$ for $n = 4,5$, $\frac{1}{2}(2n^2-5n-2)$ for $6 \le n \le 11$, 113 for $n = 12$, and $\frac{1}{3}(3n^2-8n+3)$ for $n \ge 13$;

(ii) for $m = 3$, C_n for $n = 6,7$ and $U_{n,3}$ for $n \ge 8$ are the unique graphs with the minimum Kirchhoff indices, which are equal to $\frac{n^3-n}{12}$ for $n = 6,7$ and n^2-2n-5 for $n \ge 8$;

(iii) for $m = 4$, C_8 for $n = 8$, $U(7,1,1,0)$ and C_9 for $n = 9$, $U(7,1,2,0)$ for $n = 10$, $U(6,2,3,0)$ and $U(7,1,3,0)$ for $n = 11$, $U(6,2,n-8,0)$ for $n = 12,13$, $U_{14,4}$ and $U(6,2,6,0)$ for $n = 14$, and $U_{n,m}$ for $n \ge 15$ are the unique graphs with the minimum Kirchhoff indices, which are equal to 42 for $n = 8$, 60 for $n = 9$, 79 for $n = 10$, 100 for $n = 11$, $\frac{1}{3}(3n^2-n-52)$ for $n = 12,13$, 174 for $n = 14$, and n^2-n-8 for $n \ge 15$;

(iv) for $m = 5$, $U(8,2)$ for $n = 10$, $U(7,3,n-10,0)$ for $11 \le n \le 13$, $U_{14,5}$, $U(6,2,4,1)$ and $U(7,3,4,0)$ for $n = 14$, and $U_{n,5}$ for $n \ge 15$ are the unique graphs with the minimum Kirchhoff indices, which are equal to $81\frac{7}{8}$ for $n = 10$, $\frac{1}{7}(7n^2+12n-245)$ for $11 \le n \le 13$, 185 for $n = 14$, and n^2-11 for $n \ge 15$;

(v) for $m = 6$, $U(8,4,n-12,0)$ for $n = 12,13$, $U_{14,6}$, $U(6,2,2,2)$ and $U(7,3,2,1)$ for $n = 14$, and $U_{n,6}$ for $n \ge 15$ are the unique graphs with the minimum Kirchhoff indices, which are equal to $\frac{1}{4}(4n^2+19n-262)$ for $n = 12,13$, 196 for $n = 14$, and n^2+n-14 for $n \ge 15$;

(vi) for $m = 7$, $U(7,7)$ for $n = 14$ and $U_{n,7}$ for $n \ge 15$ are the unique graphs with the minimum Kirchhoff indices, which are equal to 203 for $n = 14$ and $n^2+2n-17$ for $n \ge 15$;

(vii) for $m \ge 8$, $U_{n,m}$ for $n \ge 16$ is the unique graph with the minimum Kirchhoff index, which is equal to $n^2+nm-5n-3m+4$.

Proof. The result for $n = 2m$ follows from Theorem 3.1. Suppose that $n > 2m$. Let $G \in \mathbb{U}(n,m)$.

Case 1. $m = 2$.

Clearly, the girth of G is $3,4,5$ for $n = 5$, and $3,4$ for $n \ge 6$. Then by Lemma 2.8, we have

$$Kf(G) \;\ge\; \min\{Kf(U(3,1,1,0)), Kf(U(4,1)), Kf(C_5)\}$$

$$= \min\left\{12\frac{2}{3}, 11\frac{1}{2}, 10\right\} = 10$$

for $n = 5$, and

$$\begin{aligned} Kf(G) &\geq \min\{Kf(U(3,1,n-4,0)), Kf(U(4,1,n-5,0))\} \\ &= \min\left\{\frac{1}{3}(3n^2 - 8n + 3), \frac{1}{2}(2n^2 - 5n - 2)\right\} \end{aligned}$$

for $n \geq 6$. Thus C_5 for $n = 5$, $U(4,1,n-5,0)$ for $6 \leq n \leq 11$, $U(3,1,8,0)$ and $U(4,1,7,0)$ for $n = 12$, and $U(3,1,n-4,0)$ for $n \geq 13$ are the unique graphs in $\mathbb{U}(n,2)$ with the minimum Kirchhoff indices.

Case 2. $m = 3$.

If $G \cong C_n$, then $n = 7$, and by Eq. (3), we have $Kf(G) = 28$.

Suppose that $G \ncong C_n$. If $G_0 \cong U_{6,3}$, then by Lemma 3.2 (i), we have $Kf(G) \geq Kf(U_{n,3})$ with equality if and only if $G \cong U_{n,3}$. Suppose that $G_0 \ncong U_{6,3}$. If $Kf(G_0) \geq Kf(U_{6,3})$, then by Lemma 3.2 (ii), we have $Kf(G) > Kf(U_{n,3})$. If $Kf(G_0) < Kf(U_{6,3})$, then by Lemma 3.3 and Table 1, we assume that $G_0 = C_6$, and thus by Lemma 3.2 (i), we have $Kf(G) \geq Kf(U(6,1,n-7,0))$ with equality if and only if $G \cong U(6,1,n-7,0)$.

Therefore for $n = 7$,

$$Kf(G) \geq \min\{Kf(U_{7,3}), Kf(U(6,1)), Kf(C_7)\} = \min\left\{30, 29\frac{1}{3}, 28\right\} = 28$$

with equality if and only if $G \cong C_7$, and for $n \geq 8$,

$$\begin{aligned} Kf(G) &\geq \min\{Kf(U_{n,3}), Kf(U(6,1,n-7,0))\} \\ &= \min\left\{n^2 - 2n - 5, n^2 - \frac{7}{6}n - \frac{23}{2}\right\} = n^2 - 2n - 5 \end{aligned}$$

with equality if and only if $G \cong U_{n,3}$.

Case 3. $m = 4$.

If $G \cong C_n$, then $n = 9$, and by Eq. (3), we have $Kf(G) = 60$.

Suppose that $G \ncong C_n$. If $G_0 \cong U_{8,4}$, then by Lemma 3.2 (i), we have $Kf(G) \geq Kf(U_{n,4})$ with equality if and only if $G \cong U_{n,4}$. Suppose that $G_0 \ncong U_{8,4}$. If $Kf(G_0) \geq Kf(U_{8,4})$, then by Lemma 3.2 (ii), we have $Kf(G) > Kf(U_{n,4})$. If $Kf(G_0) < Kf(U_{8,4})$, then by Lemma 3.3 and Table 2, we assume that $G_0 = U(5,3,0,0), U(6,2,0,0), U(7,1,0,0)$ or $U(8,0,0,0)$, and thus by Lemma 3.2 (i), we have

$$\begin{aligned} Kf(G) &\geq \min\{Kf(U(5,3,n-8,0)), Kf(U(6,2,n-8,0)), \\ &\quad Kf(U(7,1,n-8,0)), Kf(U(8,0,n-8,0))\}. \end{aligned}$$

Now the result for $m = 4$ follows from Table 7 easily.

Case 4. $m = 5$.

If $G \cong C_n$, then $n = 11$, and by Eq. (3), we have $Kf(G) = 110$.

Table 7: The graphs in $\mathbb{U}(n,4)$ and their Kirchhoff indices.

Graphs	Kirchhoff indices							
	n	9	10	11	12	13	14	15
$U_{n,4}$	n^2-n-8	64	82	102	124	148	**174**	**202**
$U(5,3,n-8,0)$	$n^2-\frac{2}{5}n-15$	$62\frac{2}{5}$	81	$101\frac{3}{5}$	$124\frac{1}{5}$	$148\frac{4}{5}$	$175\frac{2}{5}$	204
$U(6,2,n-8,0)$	$n^2-\frac{1}{3}n-\frac{52}{3}$	$60\frac{2}{3}$	$79\frac{1}{3}$	**100**	$\mathbf{122\frac{2}{3}}$	$\mathbf{147\frac{1}{3}}$	**174**	$202\frac{2}{3}$
$U(7,1,n-8,0)$	n^2-21	**60**	**79**	**100**	123	148	175	204
$U(8,0,n-8,0)$	$n^2+\frac{3}{2}n-34$	$60\frac{1}{2}$	81	$103\frac{1}{2}$	128	$154\frac{1}{2}$	183	$213\frac{1}{2}$
C_9		**60**						

Suppose that $G \ncong C_n$. If $G_0 \cong U_{10,5}$, then by Lemma 3.2 (i), we have $Kf(G) \geq Kf(U_{n,5})$ with equality if and only if $G \cong U_{n,5}$. Suppose that $G_0 \ncong U_{10,5}$. If $Kf(G_0) \geq Kf(U_{10,5})$, then by Lemma 3.2 (ii), we have $Kf(G) > Kf(U_{n,5})$. If $Kf(G_0) < Kf(U_{10,5})$, then by Lemma 3.3 and Table 3, we assume that $G_0 = U(5,3,0,1)$, $U(5,5,0,0)$, $U(6,2,0,1)$, $U(6,4,0,0)$, $U(7,1,0,1)$, $U(7,3,0,0)$, $U(8,0,0,1)$, $U(8,2,0,0)$, $U(9,1,0,0)$ or $U(10,0,0,0)$, and thus by Lemma 3.2 (i), we have

$$\begin{aligned} Kf(G) \geq\ & \min\{Kf(U(5,3,n-10,1)), Kf(U(5,5,n-10,0)), Kf(U(6,2,n-10,1)), \\ & Kf(U(6,4,n-10,0)), Kf(U(7,1,n-10,1)), Kf(U(7,3,n-10,0)), \\ & Kf(U(8,0,n-10,1)), Kf(U(8,2,n-10,0)), Kf(U(9,1,n-10,0)), \\ & Kf(U(10,0,n-10,0))\}. \end{aligned}$$

Now the result for $m=5$ follows from Table 8 easily.

Case 5. $m=6$.

If $G \cong C_n$, then $n=13$, and by Eq. (3), we have $Kf(G)=182$.

Suppose that $G \ncong C_n$. If $G_0 \cong U_{12,6}$, then by Lemma 3.2 (i), we have $Kf(G) \geq Kf(U_{n,6})$ with equality if and only if $G \cong U_{n,6}$. Suppose that $G_0 \ncong U_{12,6}$. If $Kf(G_0) \geq Kf(U_{12,6})$, then by Lemma 3.2 (ii), we have $Kf(G) > Kf(U_{n,6})$. If $Kf(G_0) < Kf(U_{12,6})$, then by Lemma 3.3 and Table 4, we assume that $G_0 = U(6,2,0,2)$, $U(6,4,0,1)$, $U(6,6,0,0)$, $U(7,1,0,2)$, $U(7,3,0,1)$, $U(7,5,0,0)$, $U(8,2,0,1)$, $U(8,4,0,0)$, $U(9,3,0,0)$, $U(10,2,0,0)$ or $U(11,1,0,0)$, and thus by Lemma 3.2 (i), we have

$$\begin{aligned} Kf(G) \geq\ & \min\{Kf(U(6,2,n-12,2)), Kf(U(6,4,n-12,1)), Kf(U(6,6,n-12,0)), \\ & Kf(U(7,1,n-12,2)), Kf(U(7,3,n-12,1)), Kf(U(7,5,n-12,0)), \\ & Kf(U(8,2,n-12,1)), Kf(U(8,4,n-12,0)), Kf(U(9,3,n-12,0)), \\ & Kf(U(10,2,n-12,0)), Kf(U(11,1,n-12,0))\}. \end{aligned}$$

Now the result for $m=6$ follows from Table 9 easily.

Table 8: The graphs in $\mathbb{U}(n,5)$ and their Kirchhoff indices.

Graphs	Kirchhoff indices					
	n	11	12	13	14	15
$U_{n,5}$	n^2-11	110	133	158	**185**	**214**
$U(5,3,n-10,1)$	$n^2+\frac{3}{5}n-18$	$109\frac{3}{5}$	$133\frac{1}{5}$	$158\frac{4}{5}$	$186\frac{2}{5}$	216
$U(5,5,n-10,0)$	$n^2+2n-35$	108	133	160	189	220
$U(6,2,n-10,1)$	$n^2+\frac{2}{3}n-\frac{61}{3}$	108	$131\frac{2}{3}$	$157\frac{1}{3}$	**185**	$214\frac{2}{3}$
$U(6,4,n-10,0)$	$n^2+\frac{11}{6}n-\frac{209}{6}$	$106\frac{1}{3}$	$131\frac{1}{6}$	158	$186\frac{5}{6}$	$217\frac{2}{3}$
$U(7,1,n-10,1)$	n^2+n-24	108	132	158	186	216
$U(7,3,n-10,0)$	$n^2+\frac{12}{7}n-\frac{245}{7}$	$\mathbf{104\frac{6}{7}}$	$\mathbf{129\frac{4}{7}}$	$\mathbf{156\frac{2}{7}}$	**185**	$215\frac{5}{7}$
$U(8,0,n-10,1)$	$n^2+\frac{5}{2}n-37$	$111\frac{1}{2}$	137	$164\frac{1}{2}$	194	$225\frac{1}{2}$
$U(8,2,n-10,0)$	$n^2+\frac{19}{8}n-\frac{335}{8}$	$105\frac{1}{4}$	$130\frac{5}{8}$	158	$187\frac{3}{8}$	$218\frac{3}{4}$
$U(9,1,n-10,0)$	$n^2+\frac{10}{3}n-51$	$106\frac{2}{3}$	133	$161\frac{1}{3}$	$191\frac{2}{3}$	224
$U(10,0,n-10,0)$	$n^2+\frac{11}{2}n-\frac{145}{2}$	109	$137\frac{1}{2}$	168	$200\frac{1}{2}$	235
C_{11}		110				

Case 6. $m=7$.

If $G \cong C_n$, then $n=15$, and by Eq. (3), we have $Kf(G)=280$.

Suppose that $G \ncong C_n$. If $G_0 \cong U_{14,7}$, then by Lemma 3.2 (i), we have $Kf(G) \geq Kf(U_{n,7})$ with equality if and only if $G \cong U_{n,7}$. Suppose that $G_0 \ncong U_{14,7}$. If $Kf(G_0) \geq Kf(U_{14,7})$, then by Lemma 3.2 (ii), we have $Kf(G) > Kf(U_{n,7})$. If $Kf(G_0) < Kf(U_{14,7})$, then by Lemma 3.3 and Table 5, we assume that $G_0 = U(7,7,0,0)$, $U(8,6,0,0)$ or $U(9,5,0,0)$, and thus by Lemma 3.2 (i), we have

$$Kf(G) \;\; \geq \;\; \min\{Kf(U(7,7,n-14,0)), Kf(U(8,6,n-14,0)), Kf(U(9,5,n-14,0))\}.$$

Now the result for $m=7$ follows from Table 10 easily.

Case 7. $m \geq 8$.

If $G \cong C_n$, then $n=2m+1$, and by Eqs. (3) and (5), we have

$$Kf(C_{2m+1}) = \frac{1}{3}(2m^3+3m^2+m) > 6m^2-8m = Kf(U_{2m+1,m}).$$

Suppose that $G \ncong C_n$. By Theorem 3.1, $Kf(G_0) \geq Kf(U_{n_0,m})$. Furthermore, if $G_0 \cong U_{n_0,m}$, then by Lemma 3.2 (i), we have $Kf(G) \geq Kf(U_{n,m})$ with equality if and only if $G \cong U_{n,m}$, and if $G_0 \ncong U_{n_0,m}$, then by Lemma 3.2 (ii), we have $Kf(G) > Kf(U_{n,m})$. Then the result for $m \geq 8$ follows easily. ■

Table 9: The graphs in $\mathbb{U}(n,6)$ and their Kirchhoff indices.

Graphs	Kirchhoff indices			
	n	13	14	15
$U_{n,6}$	n^2+n-14	168	**196**	**226**
$U(6,2,n-12,2)$	$n^2+\frac{5}{3}n-\frac{70}{3}$	$167\frac{1}{3}$	**196**	$226\frac{2}{3}$
$U(6,4,n-12,1)$	$n^2+\frac{17}{6}n-\frac{227}{6}$	168	$197\frac{5}{6}$	$229\frac{2}{3}$
$U(6,6,n-12,0)$	$n^2+\frac{14}{3}n-64$	$165\frac{2}{3}$	$197\frac{1}{3}$	231
$U(7,1,n-12,2)$	$n^2+2n-27$	168	197	228
$U(7,3,n-12,1)$	$n^2+\frac{19}{7}n-38$	$166\frac{2}{7}$	**196**	$227\frac{5}{7}$
$U(7,5,n-12,0)$	$n^2+\frac{32}{7}n-63$	$165\frac{3}{7}$	197	$230\frac{4}{7}$
$U(8,2,n-12,1)$	$n^2+\frac{27}{8}n-\frac{359}{8}$	168	$198\frac{3}{8}$	$230\frac{3}{4}$
$U(8,4,n-12,0)$	$n^2+\frac{19}{4}n-\frac{131}{2}$	$\mathbf{165\frac{1}{4}}$	197	$230\frac{3}{4}$
$U(9,3,n-12,0)$	$n^2+\frac{46}{9}n-69$	$166\frac{4}{9}$	$198\frac{5}{9}$	$232\frac{2}{3}$
$U(10,2,n-12,0)$	$n^2+\frac{32}{5}n-\frac{412}{5}$	$169\frac{4}{5}$	$203\frac{1}{5}$	$238\frac{3}{5}$
$U(11,1,n-12,0)$	$n^2+8n-99$	174	209	246
C_{13}		182		

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Table 10: The graphs in $\mathbb{U}(n,7)$ and their Kirchhoff indices.

Graphs	Kirchhoff indices	
	n	15
$U_{n,7}$	$n^2+2n-17$	**238**
$U(7,7,n-14,0)$	$n^2+8n-105$	240
$U(8,6,n-14,0)$	$n^2+\frac{65}{8}n-\frac{839}{8}$	242
$U(9,5,n-14,0)$	$n^2+\frac{74}{9}n-105$	$243\frac{1}{3}$
C_{15}		280

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